The Fourier Series for Discrete-Time Periodic Signals

For a periodic signal $x(n)$ with period N, the Fourier series representation consists of N harmonically related exponential functions:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}$$

where the $\{c_k\}$ are the coefficients in the series representation.

The expression for $\{c_k\}$ can be obtained by taking the "dot-product:"

$$\sum_{n=0}^{N-1} x(n)e^{-j2\pi ln/N} = \sum_{n=0}^{N-1}\sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} e^{-j2\pi ln/N}$$

Interchanging the order of summation on the right-hand side:

$$= \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} c_k e^{j2\pi(k-l)n/N}$$

Note that:

$$\sum_{n=0}^{N-1} c_k e^{j2\pi(k-l)n/N} = \begin{cases} N, & k-l = 0, \pm N, \pm 2N, \ldots \\ 0, & otherwise \end{cases}$$

hence,

$$c_l = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi \ln N} \qquad l = 0, 1, \ldots, N-1 .$$

This is called the discrete-time Fourier series (DTFS).

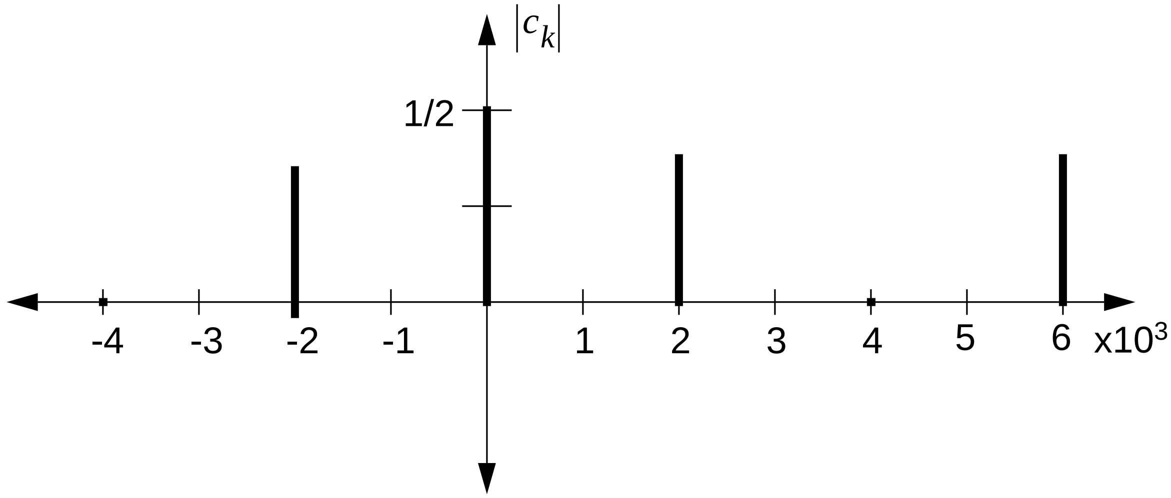
Example:

$$x(n) = \{1, 1, 0, 0\} \qquad f_s = 8000.0\ Hz$$

$$c_k = \frac{1}{4}\sum_{n=0}^{3} x(n)e^{-j2\pi kn/4} \qquad k = 0, 1, 2, 3$$

or,

$$|c_0| = \frac{1}{2}, \qquad |c_1| = \frac{\sqrt{2}}{4}, \qquad |c_2| = 0, \qquad |c_3| = \frac{\sqrt{2}}{4}$$



The Power Density Spectrum of Periodic Signals

The average power for a periodic signal was defined as:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$$

It can easily be shown that:

$$P_x = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |c_k|^2$$

The sequence $|c_k|^2$ is the distribution of power as a function of frequency and is called the power density spectrum of the periodic signal.

The energy of a sequence over a single period is given analogously as:

$$E_N = N\sum_{k=0}^{N-1} |c_k|^2 = NP_x$$

If $x(n)$ is real and periodic, we can easily show that:

$$|c_0| = |c_N|$$

$$|c_1| = |c_{N-1}| \qquad N \text{ is even}$$

$$|c_{N/2}| = |c_{N/2}|$$

$$|c_{(N-1)/2}| = |c_{(N+1)/2}| \qquad N \text{ is odd}$$

What is the significance of this result?

EXAMPLE 3.2.2 Periodic "Square-Wave" Signal

Determine the Fourier series coefficients and the power density spectrum of the periodic signal shown in Fig. 3.11.

Solution: By applying the analysis equation (3.2.8) to the signal shown in Fig. 3.11, we obtain

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} = \frac{1}{N}\sum_{n=0}^{L-1} Ae^{-j2\pi kn/N}, \qquad k = 0, 1, \ldots, N-1$$

which is a geometric summation. Now we can use (3.2.3) to simplify the summation above. Thus we obtain

$$c_k = \frac{A}{N}\sum_{n=0}^{L-1}(e^{-j2\pi k/N})^n = \begin{cases} \dfrac{AL}{N}, & k = 0 \\ \dfrac{A}{N}\dfrac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}}, & k = 1, 2, \ldots, N-1 \end{cases}$$

The last expression can be simplified further if we note that

$$\frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}} = \frac{e^{-j\pi kL/N}}{e^{-j\pi k/N}}\frac{e^{j\pi kL/N}-e^{-j\pi kL/N}}{e^{j\pi k/N}-e^{-j\pi k/N}}$$

$$= e^{-j\pi k(L-1)/N}\frac{\sin(\pi kL/N)}{\sin(\pi k/N)}$$

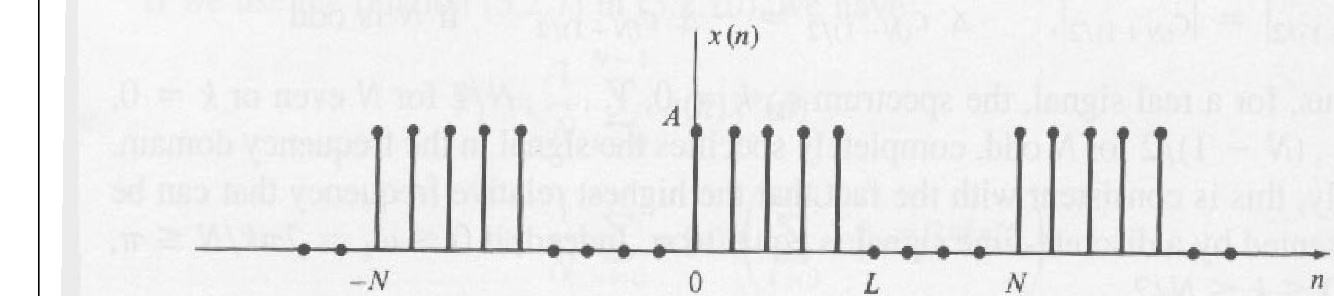


FIGURE 3.11 Discrete-time periodic square-wave signal.

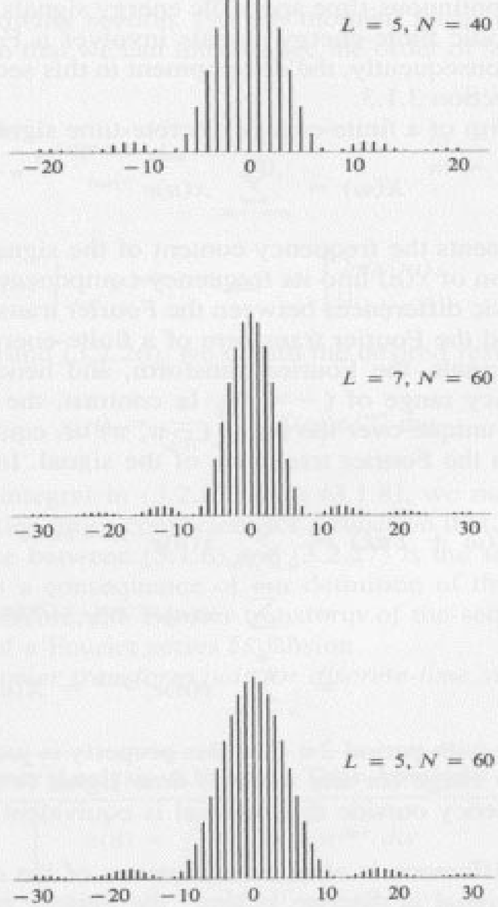


FIGURE 3.12 Plot of the power density spectrum given by (3.2.22).

Therefore,

$$c_k = \begin{cases} \dfrac{AL}{N}, & k = 0, +N, \pm 2N, \ldots \\ \dfrac{A}{N} e^{-j\pi k(L-1)/N} \dfrac{\sin(\pi kL/N)}{\sin(\pi k/N)}, & \text{otherwise} \end{cases} \tag{3.2.21}$$

The power density spectrum of this periodic signal is

$$|c_k|^2 = \begin{cases} \left(\dfrac{AL}{N}\right)^2, & k = 0, +N, \pm 2N, \ldots \\ \left(\dfrac{A}{N}\right)^2 \left(\dfrac{\sin \pi kL/N}{\sin \pi k/N}\right)^2, & \text{otherwise} \end{cases} \tag{3.2.22}$$

Figure 3.12 illustrates the plots of $|c_k|^2$ for $L = 5$ and 7, $N = 40$ and 60, and $A = 1$.

The Fourier Transform of Discrete-Time Aperiodic Signals

Define the Fourier transform of a finite-energy discrete-time signal $x(n)$ is defined as:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

Note that this is a continuous spectrum — not a line spectrum.
Also, note that it is periodic:

$$X(\omega + 2\pi k) = X(\omega)$$

The inverse of this can be shown to be:

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

This is done using a procedure analogous to the previous derivation, multiply by a basis function and integrate over a full period.

Convergence of the Fourier Transform

Define:

$$X_N(\omega) = \sum_{n=-N}^{N} x(n)e^{-j\omega n}$$

You can think of this as the short-term discrete Fourier transform (more on this later).

What happens as $N \to \infty$?

$$X_N(\omega) \to X(\omega) \qquad if \qquad \lim_{N \to \infty} \left| X(\omega) - X_N(\omega) \right| = 0$$

This is defined as uniform convergence. It can be shown that this is true if $x(n)$ is absolutely summable:

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

Some sequences are not absolutely summable, but are square summable:

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

For such sequences, the spectrum does not necessarily converge.

Example: Gibbs Phenomena

$$x(n) = \frac{\sin \omega_c n}{\pi n}, \qquad -\infty < n < \infty$$

$$X(\omega) = \begin{cases} 1 & -\omega_c < \omega < \omega_c \\ 0 & otherwise \end{cases}$$

Consider $X_N(\omega)$ (see Figure 3.14):

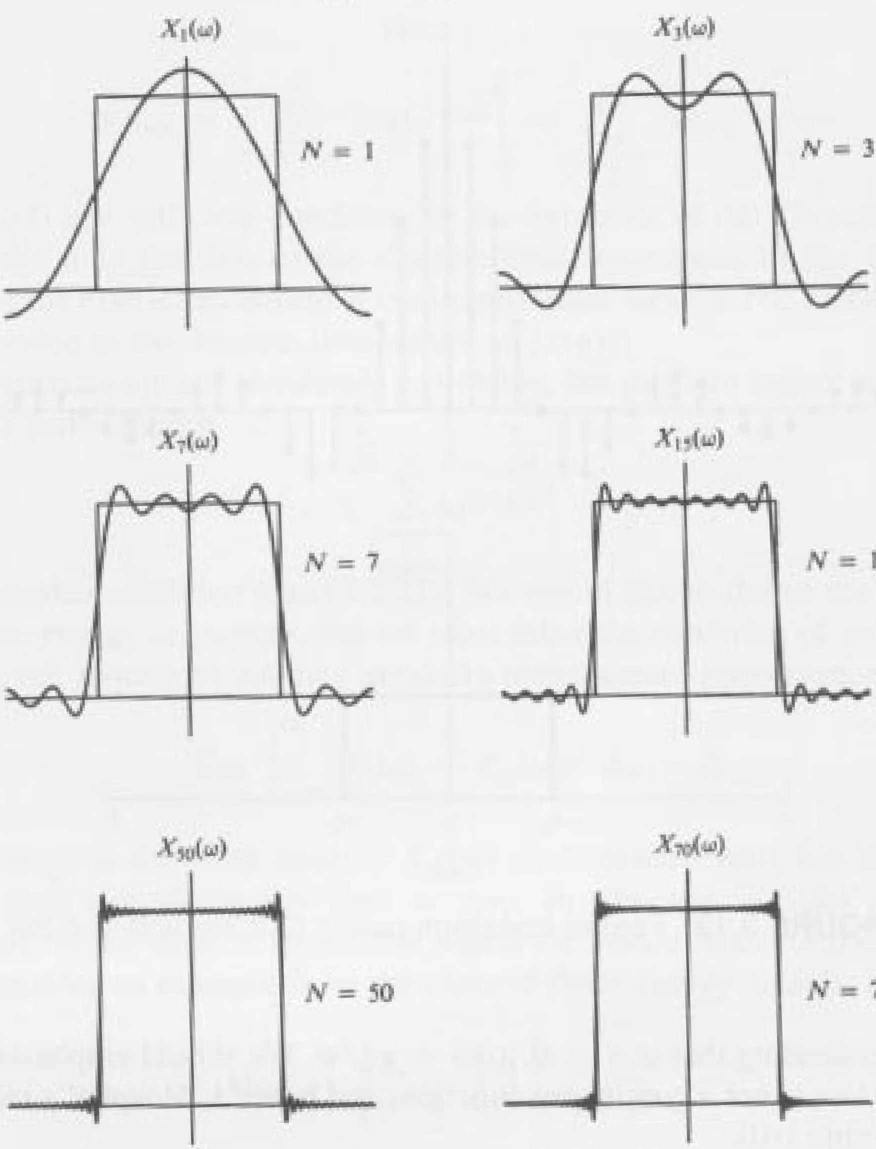


FIGURE 3.14 Illustration of convergence of the Fourier transform and the Gibbs phenomenon at the point of discontinuity.

Energy Density Spectrum of Aperiodic Signals

Recall that

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

It is easy to show that:

$$E_x = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\omega)|^2 d\omega.$$

We define the energy density spectrum of $x(n)$ as:

$$S_{xx}(\omega) = |X(\omega)|^2.$$

Suppose that $x(n)$ is real. Then, it follows that:

$$X^*(\omega) = X(-\omega),$$

or,

$|X(-\omega)| = X(\omega)$ (even symmetry) and $\angle X(-\omega) = -\angle X(\omega)$ (odd symmetry).

From this it follows that the critical frequency interval in DSP is:

$$0 \le f \le f_s / 2.$$

Summarize the differences in the spectrum of (a) a periodic square wave, (b) a single square pulse, (c) a windowed periodic square wave.