The Inverse Z-Transform By Contour Integration

The Cauchy Residue Theorem:

Let $f(z)$ be a function of the complex variable z and C be a closed path in the z-plane. If the derivative $\frac{d}{dz}f(z)$ exists on and inside the contour C and if $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{z-z_0}dz = \begin{cases} f(z_0), & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases}$$

More generally, if the $(k+1)$-order derivative of $f(z)$ exists and $f(z)$ has no poles at $z = z_0$, then

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{(z-z_0)^k}dz = \begin{cases} \frac{1}{(k-1)!}\frac{d^{(k-1)}}{dz^{k-1}}f(z)\bigg|_{z=z_0}, & \text{if } z_0 \text{ is inside } C \\ 0, & \text{if } z_0 \text{ is outside } C \end{cases}$$

the values on the right hand side are called the residues of the pole at $z = z_0$ (what is left of $f(z)$ after you remove the pole).

For $P(z) = f(z)/g(z)$, we can show:

$$\frac{1}{2\pi j}\oint_C \frac{f(z)}{g(z)}dz = \sum_{i=1}^{n} A_i(z_i)$$

where,

$$A_i(z) = (z-z_i)\frac{f(z)}{g(z)}$$

This can be applied to the inverse z-transform:

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

$$= \sum_{\substack{\text{all poles} \\ \{z_i\}\text{ inside } C}} [\text{residue of } X(z)z^{n-1} \text{ at } z = z_i]$$

$$= \sum_i (z - z_i)X(z)z^{n-1}\Big|_{z = z_i}$$

Example: Find the inverse z-transform of

$$X(z) = \frac{1}{1 - az^{-1}}, \qquad |z| > |a|$$

Using the contour integral,

$$x(n) = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1 - az^{-1}}dz = \frac{1}{2\pi j}\oint_C \frac{z^n}{z - a}dz$$

1. For $n \geq 0$, $f(z)$ has only zeros and hence no poles inside C. The only pole occurs at $z = a$. Hence,

$$x(n) = f(z_0) = a^n, \qquad n \geq 0$$

2. If $n < 0$, $f(z) = z^n$ has an nth order pole at $z = 0$. For $n = -1$, we have

$$x(-1) = \frac{1}{2\pi j}\oint_C \frac{1}{z(z - a)}dz = \frac{1}{z - a}\Big|_{z = 0} + \frac{1}{z}\Big|_{z = a} = 0$$

you can show that $x(n) = 0, n < 0$.

The Inverse Z-Transform By Power Series Expansion

Given a z-transform, expand $X(z)$ into a power series of the form:

$$X(z) = \sum_{n=-\infty}^{\infty} c_n z^{-n}$$

By uniqueness of the z-transform, $x(n) = \{c_n\}$. When $X(z)$ is rational, the process can be performed by long division.

Example:

$$X(z) = \frac{1}{1 - az^{-1}}$$

$$\begin{array}{r|l} & 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \ldots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & az^{-1} \\ & \underline{az^{-1} - a^2z^{-2}} \\ & a^2z^{-2} \\ & \underline{a^2z^{-2} - a^3z^{-3}} \\ & a^3z^{-3} \\ & \ldots \end{array}$$

$$x(n) = \{1, a^1, a^2, a^3, a^4, \ldots\}$$

Note the implication of this: a pole can be approximated by a collection of zeros. If $|a| \ll 1$, the power series can be truncated after a few terms.

Example:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{1}{(1 - z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$1 + \frac{3}{2}z^{-1} + \frac{7}{4}z^{-2} + \frac{15}{8}z^{-3} + \ldots$$

$$1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2} \;\Big\vert\; 1$$

$$1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}$$

$$\frac{3}{2}z^{-1} - \frac{1}{2}z^{-2}$$

$$\frac{3}{2}z^{-1} - \frac{9}{4}z^{-2} + \frac{3}{4}z^{-3}$$

$$\ldots$$

$$x(n) = \left\{1, \frac{3}{2}, \frac{7}{4}, \ldots\right\}$$

Consider more complicated cases:

$$X(z) = \frac{1 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} + b_4 z^{-4}}$$

The Inverse Z-Transform By Partial Fraction Expansion

In this approach, we factor $X(z)$ into a weighted sum of simpler polynomials:

$$X(z) = \alpha_1 X_1(z) + \alpha_2 X_2(z) + \ldots + \alpha_k X_k(z)$$

$x(n)$ can be easily found using the principle of linearity and superposition:

$$x(n) = \alpha_1 x_1(n) + \alpha_2 x_2(n) + \ldots + \alpha_k x_k(n)$$

Let us illustrate with a simple example:

$$X(z) = \frac{1}{1 - 1.5z^{-1} + 0.5z^{-2}} = \frac{1}{(1 - z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)}$$

$$\frac{1}{(1 - z^{-1})\left(1 - \frac{1}{2}z^{-1}\right)} = \frac{A}{1 - z^{-1}} + \frac{B}{1 - \frac{1}{2}z^{-1}}$$

$$A\left(1 - \frac{1}{2}z^{-1}\right) + B(1 - z^{-1}) = 1$$

$$for\ z^{-1} = 2, \qquad B(-1) = 1 \qquad B = -1$$

$$for\ z^{-1} = 1, \qquad A\left(\frac{1}{2}\right) = 1 \qquad A = 2$$

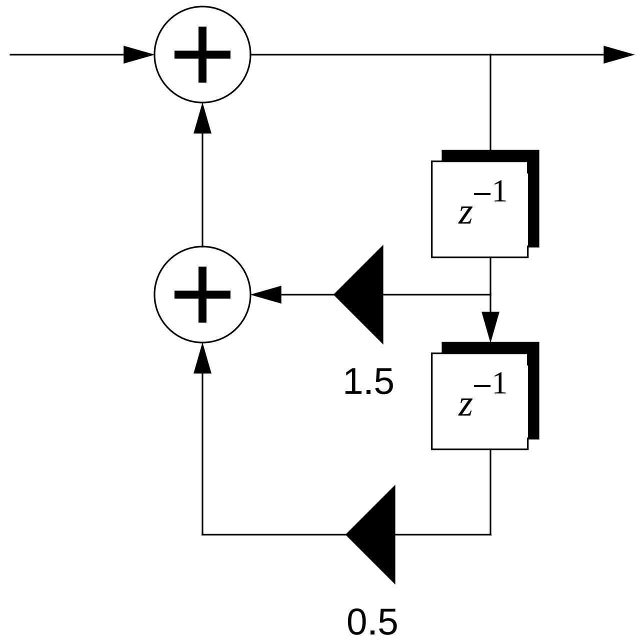
therefore,

$$X(z) = \frac{2}{1 - z^{-1}} - \frac{1}{1 - \frac{1}{2}z^{-1}}$$

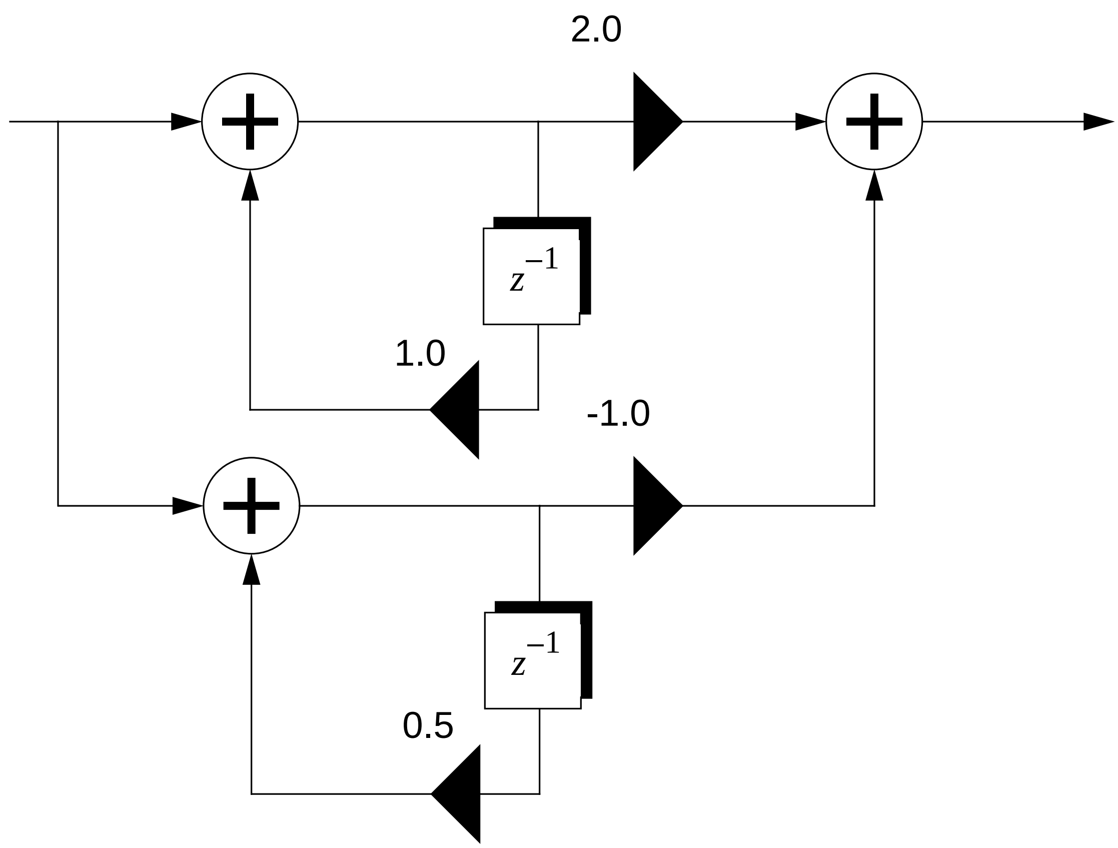
which implies that

$$x(n) = 2(1)^n u(n) - (0.5)^n u(n)$$

Note the difference in the signal flow graphs:



Direct Form Realization: 2 multiplies/accumulates
2 registers for delays



Parralel Form Realization: 4 multiplies, 3 additions
2 registers for delays

Repeated roots are a little more complicated:

$$X(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}$$

$$\frac{1}{(1+z^{-1})(1-z^{-1})^2} = \frac{A}{1+z^{-1}} + \frac{B}{1-z^{-1}} + \frac{C}{(1-z^{-1})^2}$$

$$A(1-z^{-1})^2 + B(1-z^{-1})(1+z^{-1}) + C(1+z^{-1}) = 1$$

$$for\ z^{-1} = -1, \qquad A(4) = 1 \qquad A = \frac{1}{4}$$

$$for\ z^{-1} = 1, \qquad C(2) = 1 \qquad C = \frac{1}{2}$$

$$Az^{-2} - Bz^{-2} + Cz^{-2} = 0 \qquad B = A + C = \frac{3}{4}$$

therefore,

$$X(z) = \frac{\frac{1}{4}}{1+z^{-1}} + \frac{\frac{3}{4}}{1-z^{-1}} + \frac{\frac{1}{2}}{(1-z^{-1})^2}$$

which implies that

$$x(n) = \frac{1}{4}(-1)^n u(n) + \frac{3}{4}(1)^n u(n) + \frac{1}{2}n(1)^n u(n)$$

what about $X(z) = \dfrac{1 + az^{-1} + bz^{-3}}{(1+cz^{-1})(1+dz^{-1}+ez^{-2}+fz^{-3}+gz^{-4})}$???

Derivation of z-Transform of a Sinewave

$$x(n) = (a^n \cos\omega_0 n)u(n)$$
$$= a^n\left(\frac{1}{2}\right)(e^{j\omega_0 n} + e^{-j\omega_0 n})u(n)$$

Note:

$$x(n) = a^n u(n) \Leftrightarrow X(z) = \frac{1}{1 - az^{-1}}$$

Therefore, if

$$x(n) = a^n e^{j\omega_0 n} u(n)$$
$$= (ae^{j\omega_0})^n u(n)$$

and,

$$X(z) = \frac{1}{1 - (ae^{j\omega_0})z^{-1}}$$

Thus, for the damped cosine, $x(n) = (a^n \cos\omega_0 n)u(n)$, the transform is:

$$X(z) = \frac{1/2}{1 - (ae^{j\omega_0})z^{-1}} + \frac{1/2}{1 - (ae^{-j\omega_0})z^{-1}}$$
$$= \frac{(1/2)(1 - (ae^{-j\omega_0})z^{-1}) + (1/2)(1 - (ae^{j\omega_0})z^{-1})}{(1 - (ae^{j\omega_0})z^{-1})(1 - (ae^{-j\omega_0})z^{-1})}$$
$$= \frac{1 - (a\cos\omega_0)z^{-1}}{1 - (2a\cos\omega_0)z^{-1} + a^2 z^{-2}}$$