Correlation of Discrete-Time Signals:

Consider a class of problems related to radar:

$$y(n) = \alpha x(n - D) + w(n)$$

where:

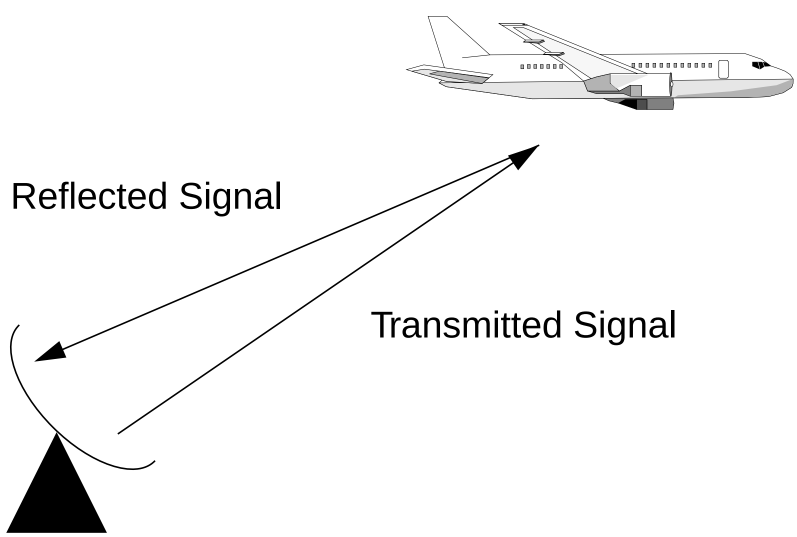
$$w(n) = \text{noise}$$

$$x(n) = \text{transmitted signal}$$

$$D = \text{round trip delay}$$

$$\alpha = \text{attenuation}$$

Note: to make things interesting, throw in a little linear filtering of the transmitted signal and a frequency shift

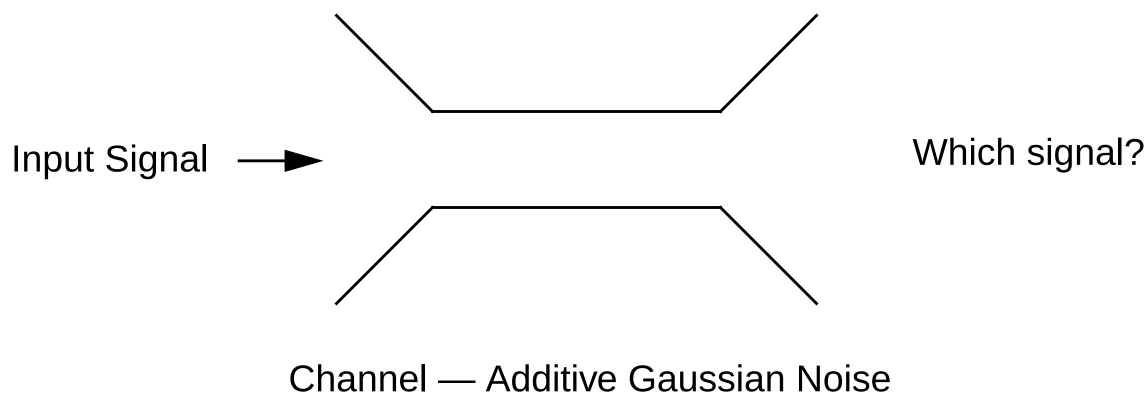


Consider a class of problems related to digital communications:

$$y(n) = x_i(n) + w(n), \qquad i = 0, 1, \qquad 0 \le n \le L - 1$$

where $x_0(n) = -x_1(n)$.

How can we optimally detect the output?



The answer to these questions involves an operation known as correlation.

Crosscorrelation and Autocorrelation of Sequences:

Define the crosscorrelation of $x(n)$ and $y(n)$ as:

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(n-l) \qquad l = 0, \pm 1, \pm 2, \ldots$$

or, equivalently,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n+l)y(n) \qquad l = 0, \pm 1, \pm 2, \ldots$$

Note that:

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n)x(n-l)$$

or,

$$r_{yx}(l) = \sum_{n=-\infty}^{\infty} y(n+l)x(n)$$

hence, we conclude that

$$r_{xy}(l) = r_{yx}(-l)$$

which means the order of crosscorrelation is more or less unimportant.

Recall our convolution operator:

The convolution of $x(n)$ and $y(-n)$ is

$$r_{xy}(l) = x(l) \otimes y(-l)$$
$$= \sum_{n=-\infty}^{\infty} x(n)\tilde{y}(l-n)$$

where $\tilde{y}(n) = y(-n)$. By simple substitution,

$$r_{xy}(l) = \sum_{n=-\infty}^{\infty} x(n)y(-l+n)$$
$$= \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$

Note for the case $y(n) = x(n)$, we define the autocorrelation as:

$$r_{xx}(l) = \sum_{n=-\infty}^{\infty} x(n)x(n-l)$$

Common alternate forms for finite duration signals (or windows of signals):

$$r_{xy}(l) = \sum_{n=l}^{N-l-1} x(n)y(n-l)$$
$$r_{xx}(l) = \sum_{n=l}^{N-l-1} x(n)x(n-l)$$

or,

$$r_{xy}(l) = \frac{1}{N-l} \sum_{n=0}^{N-l-1} x(n)y(n+l)$$
$$r_{xx}(l) = \frac{1}{N-l} \sum_{n=0}^{N-l-1} x(n)x(n+l)$$

Properties of the Autocorrelation and Crosscorrelation Sequences:

$$ax(n) + by(n-l)$$

$$\sum_{n=-\infty}^{\infty} (ax(n) + by(n-l))^2 = a^2 \sum_{n=-\infty}^{\infty} x^2(n) + b^2 \sum_{n=-\infty}^{\infty} y^2(n-l) + 2ab \sum_{n=-\infty}^{\infty} x(n)y(n-l)$$

$$= a^2 r_{xx}(0) + b^2 r_{yy}(0) + 2ab r_{xy}(l)$$

Note that $r_{xx}(0) = E_x$ and $r_{yy}(0) = E_y$, so that:

$$a^2 r_{xx}(0) + b^2 r_{yy}(0) + 2ab r_{xy}(l) \geq 0$$

divide by b^2:

$$r_{xx}(0)\left(\frac{a}{b}\right)^2 + 2r_{xy}(l)\left(\frac{a}{b}\right) + r_{yy}(0) \geq 0$$

Since the quadratic is nonnegative, it follows that the discriminant must be nonpositive:

$$4[r_{xy}^2(l) - r_{xx}(0)r_{yy}(0)] \leq 0$$

therefore,

$$\left|r_{xy}(l)\right| \leq \sqrt{r_{xx}(0)r_{yy}(0)} = \sqrt{E_x E_y}$$

and,

$$\left|r_{xx}(l)\right| \leq r_{xx}(0) = E_x$$

Define a normalized crosscorrelation:

$$\rho_{xx}(l) = \frac{r_{xx}(l)}{r_{xx}(0)}$$

and,

$$\rho_{xy}(l) = \frac{r_{xy}(l)}{\sqrt{r_{xx}(0) r_{yy}(0)}}$$

note that $|\rho_{xx}(l)| \le 1$ and $|\rho_{xy}(l)| \le 1$.

Also, recall that

$$r_{xy}(l) = r_{yx}(-l)$$

and hence,

$$r_{xx}(l) = r_{xx}(-l)$$

therefore, the autocorrelation function is an even function.

What does this imply about its Fourier transform?

Correlation of Periodic Sequences:

Let $x(n)$ and $y(n)$ be power signals. Their crosscorrelation is defined as:

$$r_{xy}(l) = \lim_{M \to \infty} \frac{1}{2M+1} \sum_{n=-M}^{M} x(n)y(n-l)$$

Note that this is equivalent to computing:

$$r_{xy}(l) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)y(n-l)$$

ONLY when N is the period!

Let $y(n) = x(n) + w(n)$. Compute the autocorrelation:

$$\begin{aligned} r_{yy}(l) &= \frac{1}{N} \sum_{n=0}^{N-1} y(n)y(n-l) \\ &= r_{xx}(l) + r_{xw}(l) + r_{wx}(l) + r_{ww}(l) \end{aligned}$$

for white, uncorrelated noise:

$$r_{yy}(l) = r_{xx}(l) + r_{ww}(l)$$

note that $r_{ww}(l) = 0 \qquad \forall l \neq 0$.

Consider Input-Output Relations:

$$y(n) = h(n) \otimes x(n)$$

$$\begin{aligned} r_{yx}(l) &= y(l) \otimes x(-l) \\ &= h(l) \otimes x(l) \otimes x(-l) \\ &= h(l) \otimes (x(l) \otimes x(-l)) \\ &= h(l) \otimes r_{xx}(l) \end{aligned}$$

Similarly,

$$r_{xy}(l) = h(-l) \otimes r_{xx}(l)$$

next,

$$\begin{aligned} r_{yy}(l) &= y(l) \otimes y(-l) \\ &= (h(l) \otimes x(l)) \otimes (h(-l) \otimes x(-l)) \\ &= (h(l) \otimes h(-l)) \otimes (x(l) \otimes x(-l)) \\ &= r_{hh}(l) \otimes r_{xx}(l) \end{aligned}$$

for $l = 0$,

$$r_{yy}(0) = \sum_{k=-\infty}^{\infty} r_{hh}(k) r_{xx}(k) .$$