The Direct z-Transform

The z-transform of a discrete-time signal is defined as the power series:

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

z is a complex variable.

The z-transform will be denoted by

$$X(z) = Z[x(n)]$$

which we will depict as:

$$x(n) \xrightarrow{z} X(z)$$

Since the z-transform is an infinite power series, it exists for only those values of z for which the series converges.

The *region of convergence* (ROC) of $X(z)$ is the set of all values of z for which $X(z)$ attains a finite value.

Examples — Finite Duration Signals

$$x_1(n) = \{1, 2, 5, 7, 0, 1\}$$

$$= \delta(n) + 2\delta(n-1) + 5\delta(n-2) + 7\delta(n-3) + \delta(n-5)$$

$$X_1(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

$$x_2(n) = \{1, 2, \mathbf{5}, 7, 0, 1\}$$

$$= \delta(n+2) + 2\delta(n+1) + 5\delta(n) + 7\delta(n-1) + \delta(n-3)$$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= 1z^{+2} + 2z^{+1} + 5 + 7z^{-1} + z^{-3}$$

$$x_3(n) = \{0, 0, 1, 2, 5, 7, 0, 1\}$$

$$= z^{-2} + 2z^{-3} + 5z^{-4} + 7z^{-5} + z^{-7}$$

For each of these, the region of convergence is the entire plane except $z = 0$ and $z = \infty$ for the second example.

Hence, finite duration sequences essentially converge everywhere.

Examples — Infinite Duration

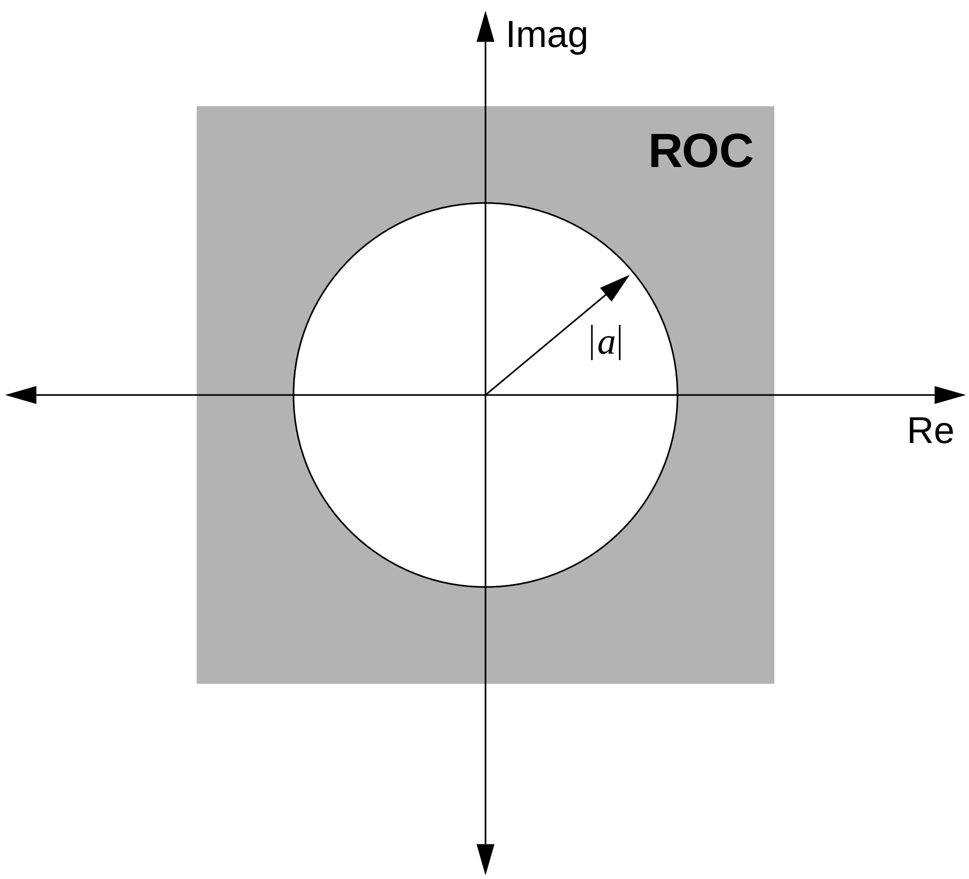
$$x(n) = a^n u(n)$$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1 - az^{-1}}$$

the ROC is the region $|z| > a$:



Note: see Example 4.1.5 for a case of a two-sided signal.

Region of Convergence

Let $z = re^{j\theta}$:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\theta n}$$

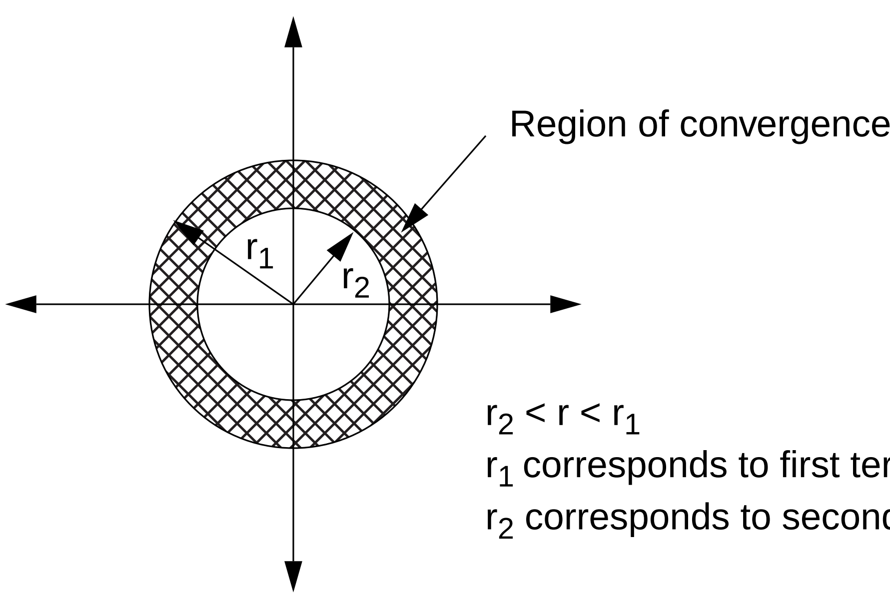
Let us examine the magnitude:

$$|X(z)| = \left|\sum_{n=-\infty}^{\infty} x(n)r^{-n}e^{-j\theta}\right|$$

$$\leq \sum_{n=-\infty}^{\infty} \left|x(n)r^{-n}e^{-j\theta n}\right| = \sum_{n=-\infty}^{\infty} \left|x(n)r^{-n}\right|$$

Hence, the z-transform converges if $x(n)r^{-n}$ is absolutely summable:

$$|X(z)| \leq \sum_{n=1}^{\infty} \left|x(-n)r^{n}\right| + \sum_{n=0}^{\infty} \left|\frac{x(n)}{r^{n}}\right|$$



$r_2 < r < r_1$

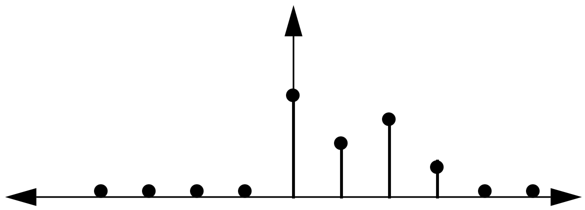
r_1 corresponds to first term

r_2 corresponds to second term

Signal	**ROC**

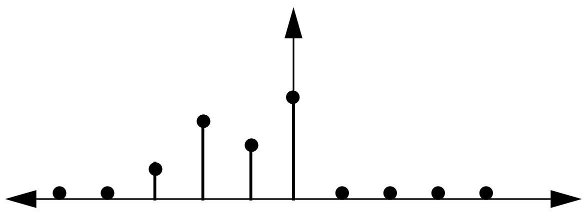
Finite Duration

Causal



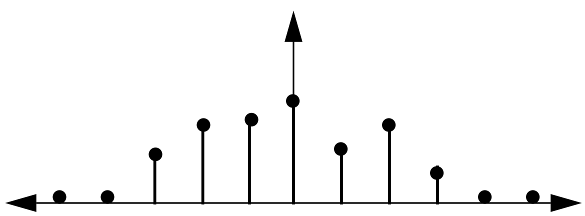
Everywhere except $z = 0$

Anticausal



Everywhere except $z = \infty$

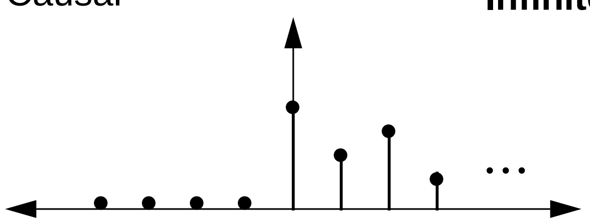
Two-sided



Entire z-plane except $z = 0$ and $z = \infty$

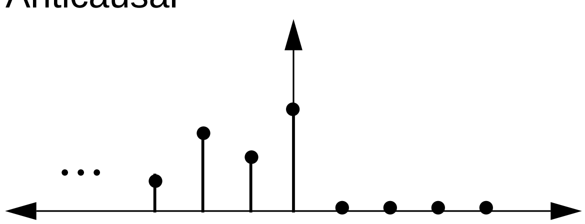
Infinite Duration

Causal



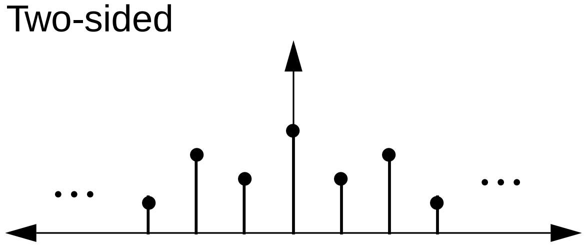
$|z| > r_2$

Anticausal



$|z| < r_1$

Two-sided



$r_2 < |z| < r_1$

The Inverse z-Transform

Recall,

$$X(z) \equiv \sum_{k=-\infty}^{\infty} x(k) z^{-k}$$

Calling upon our extensive knowledge of complex variables, we multiply by z^{-1} and integrate over a closed contour with the ROC:

$$\oint_C X(z) z^{n-1} dz = \oint_C \left(\sum_{k=-\infty}^{\infty} x(k) z^{-k} \right) z^{n-1} dz$$

$$= \sum_{k=-\infty}^{\infty} x(k) \oint_C z^{n-1-k} dz$$

we can invoke the Cauchy integral theorem:

$$\frac{1}{2\pi j} \oint_C z^{n-1-k} dz = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$$

Hence,

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

(Possibly the most useless equation in this course!)

Relationship of the Fourier Transform to the z-Transform

Recall,

$$X(z) \equiv \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Let $z = re^{j\omega}$, then:

$$X(z)\big|_{z = re^{j\omega}} = \sum_{n=-\infty}^{\infty} [x(n) r^{-n}] z^{-j\omega n}$$

If $X(z)$ converges for $|z| = 1$, then:

$$X(z)\big|_{z = e^{j\omega}} \equiv X(\omega) = \sum_{n=-\infty}^{\infty} x(n) z^{-j\omega n}$$

The Fourier transform may be viewed as the z-transform evaluated around the unit circle.

Note that there are sequences whose z-transform does not exist, yet their Fourier transform does (the ideal lowpass filter is the most prominent example).

Therefore, what are the relationships between the Fourier series, Fourier transform, and z-transform?

(Consider the case of a periodic signal for which the interval of summation is not equal to the period of the signal.)

The (Complex) Cepstrum

Consider a sequence $\{x(n)\}$ having a z-transform $X(z)$. The complex cepstrum of the sequence $\{x(n)\}$ is defined as the sequence $\{c_x(n)\}$, which is the inverse z-transform of $C_x(z)$, where

$$C_x(z) = \ln X(z)$$

The complex cepstrum, in the region of convergence of the z-transform, may be represented by the Laurent series

$$C_x(z) = \ln X(z) = \sum_{n=-\infty}^{\infty} c_x(n) z^{-n}$$

where $\{c_x(n)\}$ is the inverse z-transform of $\ln X(z)$:

$$c_x(n) = \frac{1}{2\pi j}\oint_C \ln X(z) z^{n-1} dz$$

We can write the complex cepstrum in terms of the Fourier transform by noting that:

$$C_x(\omega) = \ln X(\omega) = \sum_{n=-\infty}^{\infty} c_x(n) e^{-j\omega n}$$

from the inverse transform, we can compute $\{c_x(n)\}$ as:

$$c_x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\log|X(\omega)| + j\theta(\omega)) e^{j\omega n} d\omega$$

and,

$$Re[c_x(n)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} (\log|X(\omega)|) e^{j\omega n} d\omega$$

Properties of the z-Transform

Property	Time Domain	z-Domain
Notation	$x(n)$	$X(z)$
	$x_1(n)$	$X_1(z)$
	$x_2(n)$	$X_2(z)$
Linearity and Superposition	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$
Time-Shifting	$x(n-k)$	$z^{-k}X(z)$
Scaling in the z-domain	$a^nx(n)$	$X(a^{-1}z)$
Time reversal	$x(-n)$	$X(z^{-1})$
Conjugation	$x^*(n)$	$X^*(z^*)$
Real part	$Re[x(n)]$	$\frac{1}{2}[X(z) + X^*(z^*)]$
Imag part	$Imag[x(n)]$	$\frac{1}{2j}[X(z) - X^*(z^*)]$
Differentiation in the z-domain	$nx(n)$	$-z\frac{d}{dz}X(z)$
Convolution	$x_1(n) \otimes x_2(n)$	$X_1(z)X_2(z)$
Correlation	$r_{x_1x_2}(l) = x_1(l) \otimes x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$

Properties of the z-Transform (cont.)

Property	Time Domain	z-Domain
Initial Value theor.	if $x(n)$ is causal	$x(0) = \lim_{z \to \infty} X(z)$
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*\left(\frac{1}{v^*}\right)v^{-1}dv$	

Selected proofs:

(1) Linearity:

$$Z[a_1x_1(n) + a_2x_2(n)] = \sum_{n=-\infty}^{\infty} (a_1x_1(n) + a_2x_2(n))z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1x_1(n)z^{-n} + \sum_{n=-\infty}^{\infty} a_2x_2(n)z^{-n}$$

$$= a_1\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= a_1X_1(z) + a_2X_2(z)$$

(2) Convolution:

$$x(n) = \sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty}\left[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)\right]z^{-n}$$

Interchange the order of summation:

$$= \sum_{k=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x_1(k)x_2(n-k)z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)\sum_{n=-\infty}^{\infty} x_2(n-k)z^{-n}$$

Using the time-shifting property:

$$= \sum_{k=-\infty}^{\infty} x_1(k)z^{-k}X_2(z)$$

$$= X_2(z)\sum_{k=-\infty}^{\infty} x_1(k)z^{-k}$$

$$= X_2(z)X_1(z) = X_1(z)X_2(z)$$

Example:

$$x(n) = [3(2^n) - 4(3^n)]u(n)$$

$$X(z) = 3Z[2^n] - 4Z[3^n]$$

or,

$$X(z) = \frac{3}{1-2z^{-1}} - \frac{4}{1-3z^{-1}}$$

The region of convergence for the second term is a subset of the region of convergence for the first term. Hence,

$$\text{ROC: } |z| > 3$$

Example:

$$x(n) = \cos(\omega_0 n)u(n)$$

From Euler's identity:

$$x(n) = \frac{1}{2}e^{j\omega_0 n}u(n) + \frac{1}{2}e^{-j\omega_0 n}u(n)$$

$$X(z) = \frac{1}{2}\frac{1}{1-e^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-e^{-j\omega_0}z^{-1}}$$

Example:

$$x(n) = a^n\cos(\omega_0 n)u(n)$$

$$X(z) = X(a^{-1}z)$$

or,

$$X(z) = \frac{1}{2}\frac{1}{1-ae^{j\omega_0}z^{-1}} + \frac{1}{2}\frac{1}{1-ae^{-j\omega_0}z^{-1}}$$

Example:

$$x_1(n) = \{1, -2, 1\}$$
$$x_2(n) = \{1, 1, 1, 1, 1, 1\} = u(n) - u(n-6)$$

$$X_1(z) = 1 - 2z^{-1} + z^{-2}$$
$$X_2(z) = 1 + z^{-1} + z^{-2} + z^{-3} + z^{-4} + z^{-5}$$

therefore,

$$\begin{aligned} Y(z) &= X_1(z)X_2(z) \\ &= 1 - z^{-1} - z^{-6} + z^{-7} \\ x(n) &= \{1, -1, 0, 0, 0, 0, -1, 1\} \end{aligned}$$

Example:

$$x(n) = a^n u(n)$$

Compute the autocorrelation function:

$$\begin{aligned} R_{xx}(z) &= X(z)X(z^{-1}) \\ &= \frac{1}{1 - az^{-1}} \frac{1}{1 - az} \end{aligned}$$

$$r_{xx}(l) = \frac{1}{1 - a^2} a^{|l|}$$

How do we find the energy density spectrum?