

Design of IIR Filters From Analog Filters

We would like to leverage a large existing body of analog filter design theory (keeps those old analog people employed!). An analog filter may be described in three different ways:

(1) By its system function:

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^M \beta_k s^k}{\sum_{k=0}^N \alpha_k s^k}$$

where $\{\alpha_k\}$ and $\{\beta_k\}$ are the filter coefficients.

(2) By a Laplace transform of the impulse response:

$$H_a(s) = \int_{-\infty}^{\infty} h(t) e^{-st} dt$$

(3) By a linear constant-coefficient differential equation:

$$\sum_{k=0}^N \alpha_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^M \beta_k \frac{d^k}{dt^k} x(t)$$

Each characterization results in a different digital filter.

Our conversion technique should have the following properties:

1. The $j\Omega$ axis in the s -plane should map onto the unit circle (so that the frequency response is preserved).
2. The left-half plane (LHP) of the s -plane should map into the inside of the unit circle in the z -plane (so that stability is preserved).

Design By The Approximation of Derivatives

Approach: approximate the derivative by a similar difference equation.

Backward Difference:

$$\frac{d}{dt}y(t) = \frac{y(nT) - y(nT - T)}{T} = \frac{y(n) - y(n-1)}{T}$$

We can extend this to the second derivative:

$$\begin{aligned} \left. \frac{d^2}{dt^2}y(t) \right|_{t=nT} &= \frac{d}{dt} \left[\left. \frac{d}{dt}y(t) \right]_{t=nT} \\ &= \frac{d}{dt} \left[\frac{y(nT) - y(nT - T)}{T} \right] \\ &= \frac{\left(\frac{y(nT) - y(nT - T)}{T} \right) - \left(\frac{y(nT - T) - y(nT - T - T)}{T} \right)}{T} \\ &= \frac{y(n) - 2y(n-1) + y(n-2)}{T^2} \end{aligned}$$

This implies:

$$s^2 = \left(\frac{1 - z^{-1}}{T} \right)^2$$

or, in general,

$$s^k = \left(\frac{1 - z^{-1}}{T} \right)^k$$

Since $H(s) = s$ for a differentiator, and $H(z) = \frac{1 - z^{-1}}{T}$, the frequency domain equivalent of this approximation is:

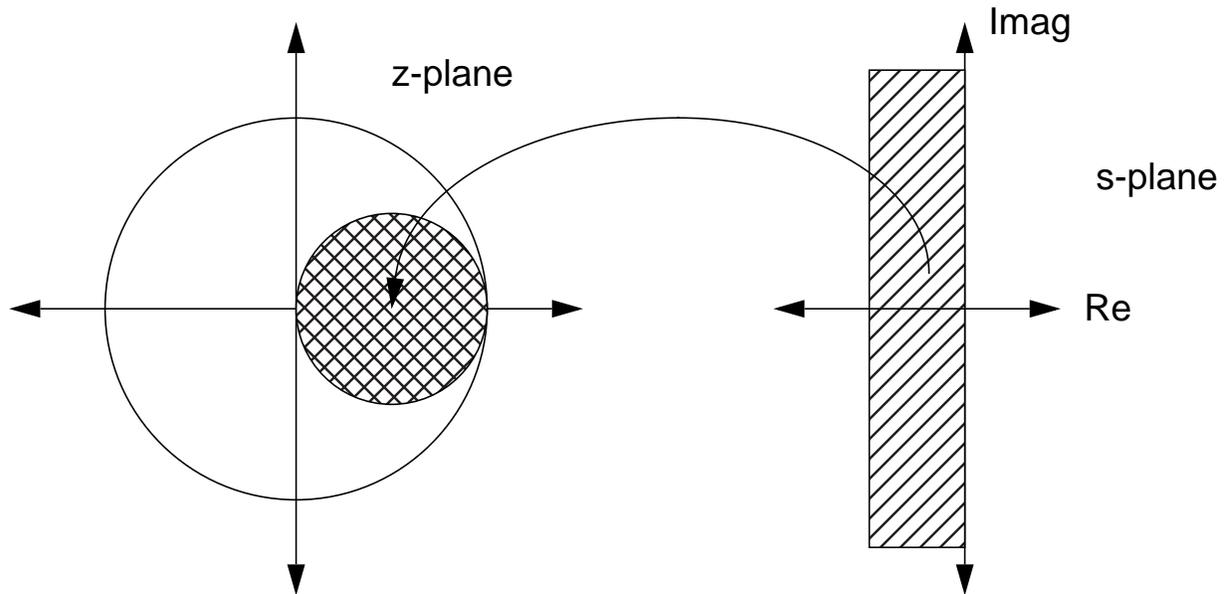
$$H(z) = H(s) \Big|_{s = \frac{1 - z^{-1}}{T}}$$

This technique is commonly used to solve differential equations numerically on a computer.

Design By The Approximation of Derivatives (Cont.)

What are the implications of this procedure?

Think of the substitution as a complex mapping:



Because the poles and zeros of the filter are pushed into a small region within the unit circle, this technique is used mainly to design lowpass filters.

This mapping distorts the frequency response of the filter (compare the analog and digital filter frequency responses). This is called “warping.”

Better approximations can be achieved by higher-order approximations for the derivatives:

$$\left. \frac{d}{dt}y(t) \right|_{t=nT} = \frac{1}{T} \sum_{k=1}^L \alpha_k \frac{y(nT+kT) - y(nT-kT)}{T}$$

Design by Impulse Invariance

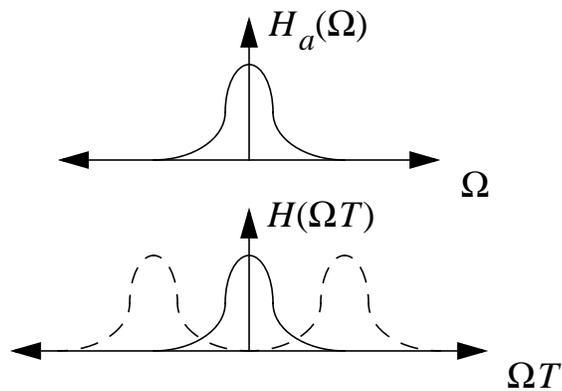
The basic idea is to match system impulse responses:

$$h(n) = h_a(t) \Big|_{t = nT}$$

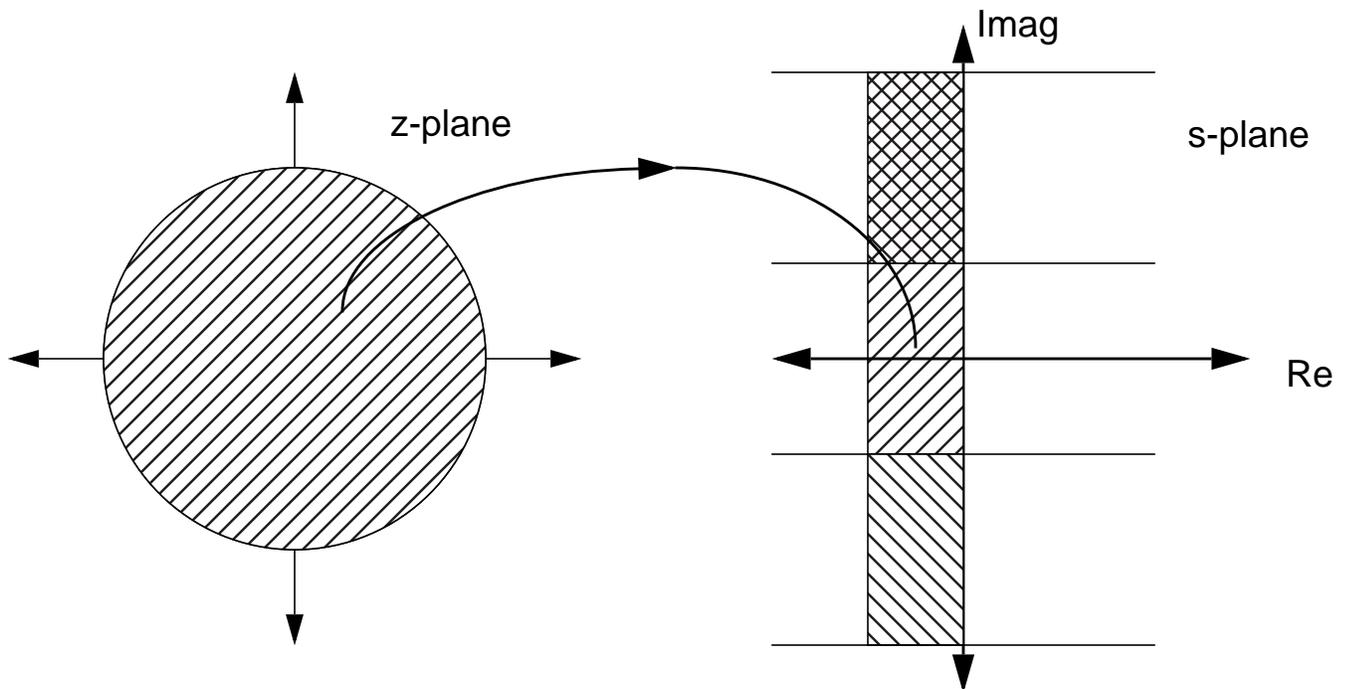
Recall the implications of this:

$$H(f) = F_s \sum_{k=-\infty}^{\infty} H_a[(f - k)F_s]$$

Note that this implies aliasing in the frequency domain:



The frequency mapping can be described as follows:



We can develop the mapping technique directly in the z -domain by writing the system function as a partial fractions expansion:

$$H_a(s) = \sum_{k=1}^N \frac{c_k}{s - p_k}$$

The inverse Laplace transform gives the impulse response:

$$h_a(t) = \sum_{k=1}^N c_k e^{p_k t}, \quad t \geq 0$$

We can sample the impulse response:

$$h(n) = h_a(nT) = \sum_{k=1}^N c_k e^{p_k nT}$$

From this we can find the z -transform:

$$H(z) = \sum_{n=0}^{\infty} h(n) z^{-n} = \sum_{n=0}^{\infty} \left(\sum_{k=1}^N c_k e^{p_k nT} \right) z^{-n}$$

After some straightforward simplification:

$$H(z) = \sum_{k=1}^N \frac{c_k}{1 - e^{p_k T} z^{-1}}$$

Hence, the digital filter has poles at $z_k = e^{p_k T}$.

The design technique is as follows:

- (1) Perform a partial fractions expansion on $H_a(s)$.
- (2) Transform each pole into its z -transform equivalent.
- (3) Combine the terms into a single polynomial.

Again, note that the resulting frequency response in the z -domain will not be the same as the “analog” frequency response.

The Bilinear Transform

Consider an analog filter:

$$H(s) = \frac{b}{s + a}$$

This system can be characterized by a differential equation:

$$\frac{d}{dt}y(t) + ay(t) = bx(t)$$

Suppose we approximate the integral rather than the derivative:

$$y(t) = \int_{t_0}^t y'(\tau) d\tau + y(t_0)$$

We can approximate this integral using a trapezoidal formula:

$$y(nT) = \frac{T}{2}[y'(nT) + y'(nT - T)] + y(nT - T)$$

From the differential equation above, we can substitute for $y'(nT)$:

$$y'(nT) = -ay(nT) + bx(nT)$$

We can substitute this result into the trapezoidal rule, and write:

$$\left(1 + \frac{aT}{2}\right)y(n) - \left(1 - \frac{aT}{2}\right)y(n-1) = \frac{bT}{2}[x(n) + x(n-1)]$$

The z -transform of this equation gives:

$$\left(1 + \frac{aT}{2}\right)Y(z) - \left(1 - \frac{aT}{2}\right)z^{-1}Y(z) = \frac{bT}{2}(1 + z^{-1})X(z)$$

This can be simplified to:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b}{\frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) + a}$$

Clearly, the mapping is as follows:

$$H(z) = H(s) \Big|_{s = \frac{2}{T}\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right)}$$

This mapping is known as the bilinear transformation.

To investigate the properties of this mapping, let $z = re^{j\omega}$ and $s = \sigma + j\Omega$:

$$\sigma + j\Omega = \frac{2}{T} \left(\frac{re^{j\omega} - 1}{re^{j\omega} + 1} \right)$$

This can be simplified to:

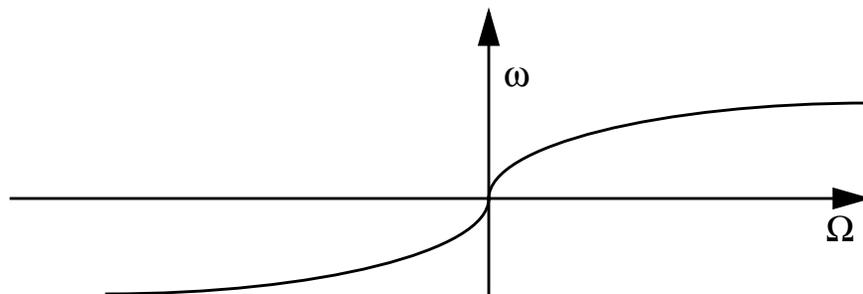
$$\sigma = \frac{2}{T} \frac{r^2 - 1}{1 + r^2 + 2r \cos \omega}$$
$$\Omega = \frac{2}{T} \left(\frac{2r \sin \omega}{1 + r^2 + 2r \cos \omega} \right)$$

If $r < 1$ then $\sigma < 0$; if $r > 1$ then $\sigma > 0$; what does this imply?

For $r = 1$, $\sigma = 0$, and

$$\Omega = \frac{2}{T} \frac{\sin \omega}{1 + \cos \omega} = \frac{2}{T} \tan \frac{\omega}{2}$$
$$\omega = 2 \tan^{-1} \frac{\Omega T}{2}$$

A plot of ω vs. Ω reveals:



The frequency compression we observe is known as a “frequency warping.”

What does this imply about a filter design methodology using the bilinear transformation?