Does a differential equation describe a linear system?

Example 1: Consider the following system:

$$\frac{d}{dt}y(t) + ty(t) = x(t)$$

Define $y_1(t)$ as the output due to $x_1(t)$, and $y_2(t)$ as the output due to $x_2(t)$. Let's compute the output to these:

$x_1(t)$: $$\frac{d}{dt}y_1(t) + ty_1(t) = x_1(t) \quad (1)$$

$x_2(t)$: $$\frac{d}{dt}y_2(t) + ty_2(t) = x_2(t) \quad (2)$$

$\alpha x_1(t) + \beta x_2(t)$: Multiply (1) by α and (2) by β, and add (1) and (2):

$$\alpha\left(\frac{d}{dt}y_1(t) + ty_1(t)\right) + \beta\left(\frac{d}{dt}y_2(t) + ty_2(t)\right) = \alpha x_1(t) + \beta x_2(t)$$

Rearranging terms:

$$\frac{d}{dt}(\alpha y_1(t) + \beta y_2(t)) + t(\alpha y_1(t) + \beta y_2(t)) = \alpha x_1(t) + \beta x_2(t) \quad (3)$$

If we define $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ and $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, we see that (3) can be rewritten as:

$$\frac{d}{dt}y_3(t) + ty_3(t) = x_3(t)$$

Hence, it is linear system.

Example 2: Consider the following system:

$$\frac{d}{dt}y(t) = x^2(t)$$

Define $y_1(t)$ as the output due to $x_1(t)$, and $y_2(t)$ as the output due to $x_2(t)$. Let's compute the output to these:

$x_1(t)$: $$\frac{d}{dt}y_1(t) = x_1^2(t) \tag{1}$$

$x_2(t)$: $$\frac{d}{dt}y_2(t) = x_2^2(t) \tag{2}$$

$\alpha x_1(t) + \beta x_2(t)$: Multiply (1) by α and (2) by β, and add (1) and (2):

$$\alpha\left(\frac{d}{dt}y_1(t)\right) + \beta\left(\frac{d}{dt}y_2(t)\right) = \alpha^2 x_1^2(t) + \beta^2 x_2^2(t)$$

Rearranging terms:

$$\frac{d}{dt}(\alpha y_1(t) + \beta y_2(t)) = \alpha^2 x_1^2(t) + \beta^2 x_2^2(t) \tag{3}$$

If we define $y_3(t) = \alpha y_1(t) + \beta y_2(t)$ and $x_3(t) = \alpha x_1(t) + \beta x_2(t)$, we see that (3) cannot be rewritten as:

$$\frac{d}{dt}y_3(t) = x_3^2(t)$$

because

$$x_3^2(t) = (\alpha x_1(t) + \beta x_2(t))^2 \neq \alpha^2 x_1^2(t) + \beta^2 x_2^2(t)$$

Therefore, the system is not linear!