Name: Richard Duncan's Exam 1 Solution

Problem	Points	Score
1a	10	
1b	10	
1c	10	
1d	10	
2a	10	
2b	10	
2c	10	
2d	10	
3a	10	
3b	10	
Total	100	

Notes:

1. The exam is closed books/closed notes - except for one page of notes.

2. Please show ALL work. Incorrect answers with no supporting explanations or work will be given no partial credit.

3. Please indicate clearly your answer to the problem. If I can't read it (and I am the judge of legibility), it is wrong. If I can't follow your solution (and I get lost easily), it is wrong. All things being equal, neat and legible work will get the higher grade:)

Problem No. 1: Modeling Concepts

(a) Prove whether the signal $x(t) = te^{-\alpha t}u(t)$ is an energy signal or power signal.

$$E \equiv \lim_{T\to\infty}\int_{-T}^{T}|x(t)|^2dt = \lim_{T\to\infty}\int_{-T}^{T}\left|te^{-\alpha t}u(t)\right|^2dt = \lim_{T\to\infty}\int_{0}^{T}\left|te^{-\alpha t}\right|^2dt$$

$$= \lim_{T\to\infty}\int_0^T t^2e^{-2\alpha t}dt = \lim_{T\to\infty} t^2\frac{e^{-2\alpha t}}{-2\alpha} - \int_0^T\left(\frac{e^{-2\alpha t}}{-2\alpha}\right)2t\,dt = \lim_{T\to\infty} t^2\frac{e^{-2\alpha t}}{-2\alpha} + \frac{1}{\alpha}\int_0^T te^{-2\alpha t}dt$$

$$= \lim_{T\to\infty} t^2\frac{e^{-2\alpha t}}{-2\alpha} + \frac{1}{\alpha}\left(t\frac{e^{-2\alpha t}}{-2\alpha} - \int_0^T\frac{e^{-2\alpha t}}{-2\alpha}\right) = \lim_{T\to\infty} t^2\frac{e^{-2\alpha t}}{-2\alpha} - t\frac{e^{-2\alpha t}}{2\alpha^2} + \frac{1}{2\alpha^2}\cdot\left(\frac{e^{-2\alpha t}}{-2\alpha}\right)\Bigg|_0^T$$

$$= \lim_{T\to\infty} t^2\frac{e^{-2\alpha t}}{-2\alpha} - t\frac{e^{-2\alpha t}}{2\alpha^2} - \frac{e^{-2\alpha t}}{4\alpha^2}\Bigg|_0^T = \lim_{T\to\infty}\left(T^2\frac{e^{-2\alpha T}}{-2\alpha} - T\frac{e^{-2\alpha T}}{2\alpha^2} - \frac{e^{-2\alpha T}}{4\alpha^2}\right) - \left(0 - 0 - \frac{1}{4\alpha^2}\right)$$

$$= \lim_{T\to\infty} T^2\frac{e^{-2\alpha T}}{-2\alpha} + \lim_{T\to\infty} -T\frac{e^{-2\alpha T}}{2\alpha^2} + \lim_{T\to\infty}\frac{-e^{-2\alpha T}}{4\alpha^2} + \lim_{T\to\infty}\frac{1}{4\alpha^2} = \begin{cases} \dfrac{1}{4\alpha^2} & ,\alpha > 0 \\ \infty & ,\alpha \le 0 \end{cases}$$

Therefore, for $0 < \alpha < \infty$, $x(t)$ has a finite value, thus $x(t)$ is an energy signal in this range.

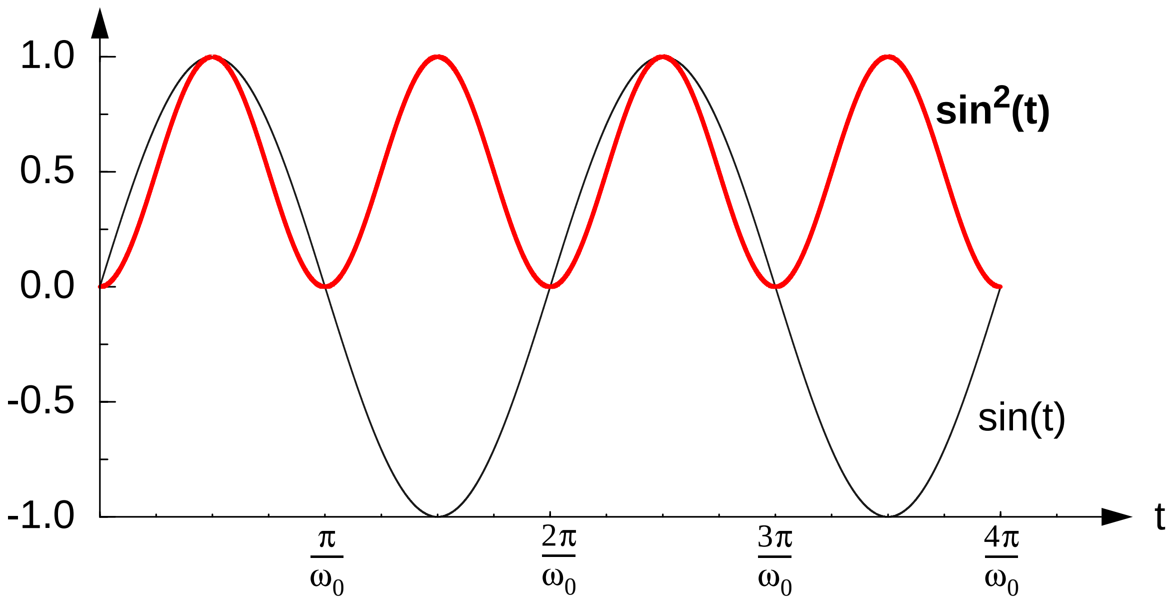
For $\alpha = 0$, this signal is neither an energy or a power signal, since by the first calculations it is not a energy signal, and:

$$P \equiv \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|te^{0}u(t)\right|^2dt = \lim_{T\to\infty}\frac{1}{2T}\int_0^T t^2dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\frac{t^3}{3}\Bigg|_0^T = \lim_{T\to\infty}\frac{1}{2T}\left[\frac{T^3}{3} - 0\right] = \lim_{T\to\infty}\frac{T^2}{6} = \infty$$

Likewise, For $\alpha < 0$, the signal is not an energy or a power signal, since $\lim_{T\to\infty} T^n e^{aT}\Big|_{a>0} = \infty$ for all powers of n, as can be proven through repeated application of Guillaume l'Hôpital's rule.

(b) Is the signal $x(t) = \sin^2\omega_0 t$ periodic? If so, what is its period? If not, explain.



Yes, this signal is periodic.

This can be proven graphically (above) as well as mathematically from the definition of a periodic function. A function is said to be periodic if $x(t) = x(t+\tau)$, for some τ. In the case of the sine function, $x(t) = \sin(\omega_0 t) = \sin(\omega_0 t + \tau_1)$, where the period is obviously $\tau_1 = \frac{2\pi}{\omega_0}$. For the sine squared function, the value always evaluates above zero (positive), hence the period is half that of the general sine function, so $\tau = \frac{\pi}{\boldsymbol{\omega_0}}$.

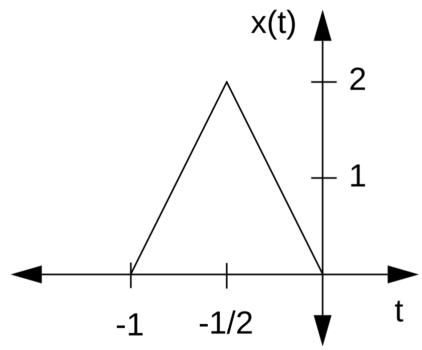
This can also be proven through the application of trigonometric identity, specifically

$$\sin^2\omega_0 t = \frac{1}{2}(1-\cos 2\omega_0 t) = \frac{1}{2}\left(1-\cos 2\omega_0\left(t+\frac{\pi}{\boldsymbol{\omega_0}}\right)\right) = \sin^2\omega_0\left(t+\frac{\pi}{\boldsymbol{\omega_0}}\right),$$

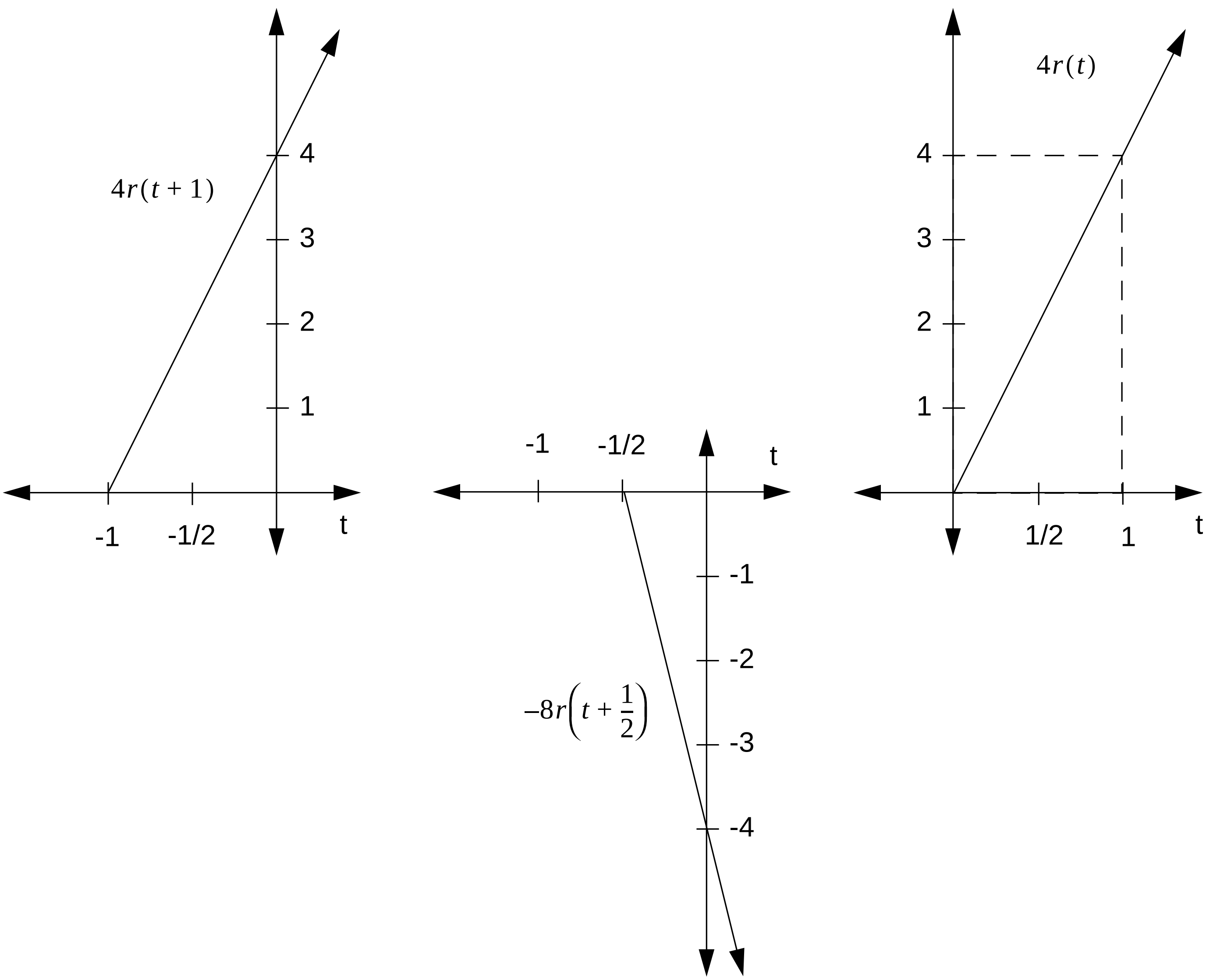
since

$$\cos(\omega_0 t) = \cos\left(\omega_0\left(t+\frac{\mathbf{2\pi}}{\boldsymbol{\omega_0}}\right)\right) = \cos(\omega_0 t + 2\pi).$$

(c) Write the following signal in terms of a weighted sum of a combination of one or all of the following functions: $\delta(t), u(t), r(t)$.



One possible solution is the sum of three ramp functions.



$$\therefore, x(t) = 4r(t+1) - 8r\left(t+\frac{1}{2}\right) + 4r(t)$$

(d) Compute the energy value of the signal in (c).

The general form of the energy equation is

$$E \equiv \lim_{T \to \infty} \int_{-T}^{T} |x(t)|^2 dt .$$

Since we know the range of this function [-1,0], we can simplify the expression to:

$$E = \int_{-1}^{0} |x(t)|^2 dt .$$

Through linearity, we can break up the function into two portions, hence,

$$E = \int_{-1}^{-\frac{1}{2}} |x(t)|^2 dt + \int_{-\frac{1}{2}}^{0} |x(t)|^2 dt .$$

Observing that the signal is symmetric about $t = -\frac{1}{2}$, we can obtain:

$$E = 2\int_{-1}^{-\frac{1}{2}} |x(t)|^2 dt .$$

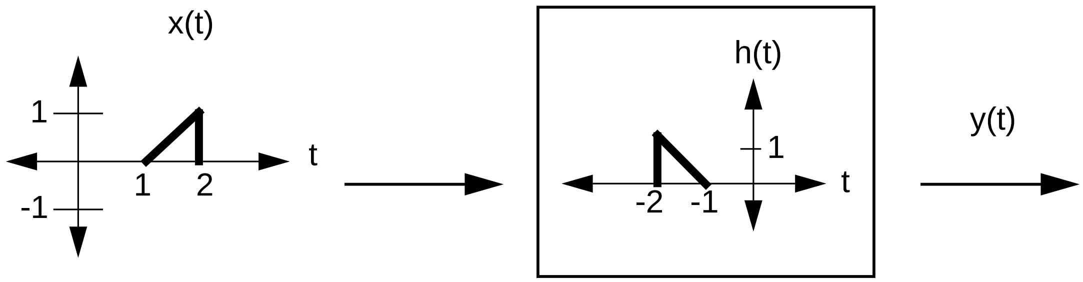
For ease of calculation, the signal and the limits of integration can be shifted,

$$E = 2\left[\int_{0}^{\frac{1}{2}} (4t)^2 dt\right] = 32\int_{0}^{\frac{1}{2}} t^2 dt = \frac{32}{3}t^3\Big|_0^{\frac{1}{2}} = \left(\frac{32}{3}\right)\left(\frac{1}{8}\right) = \frac{4}{3}.$$

Therefore, the total energy of the signal is $\frac{4}{3}$ Joules.

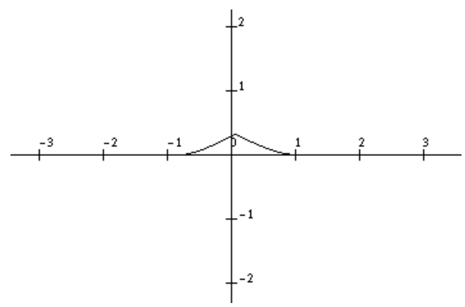
Problem No. 2: Time-Domain Solutions

Consider the signal and system (completely described by its impulse response):



(a) Compute and plot output, $y(t)$, for the system shown above.

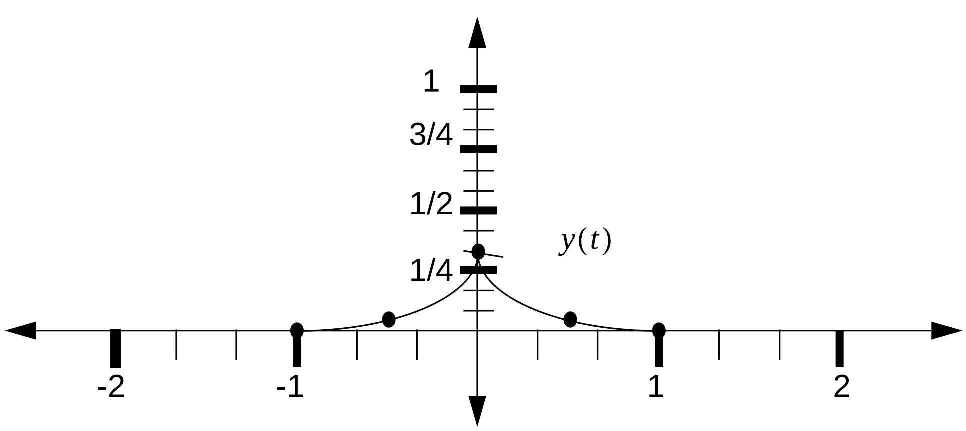
As a starting point, the output of Eric Wheeler's Convolution Tool for this signal and system is shown at right.

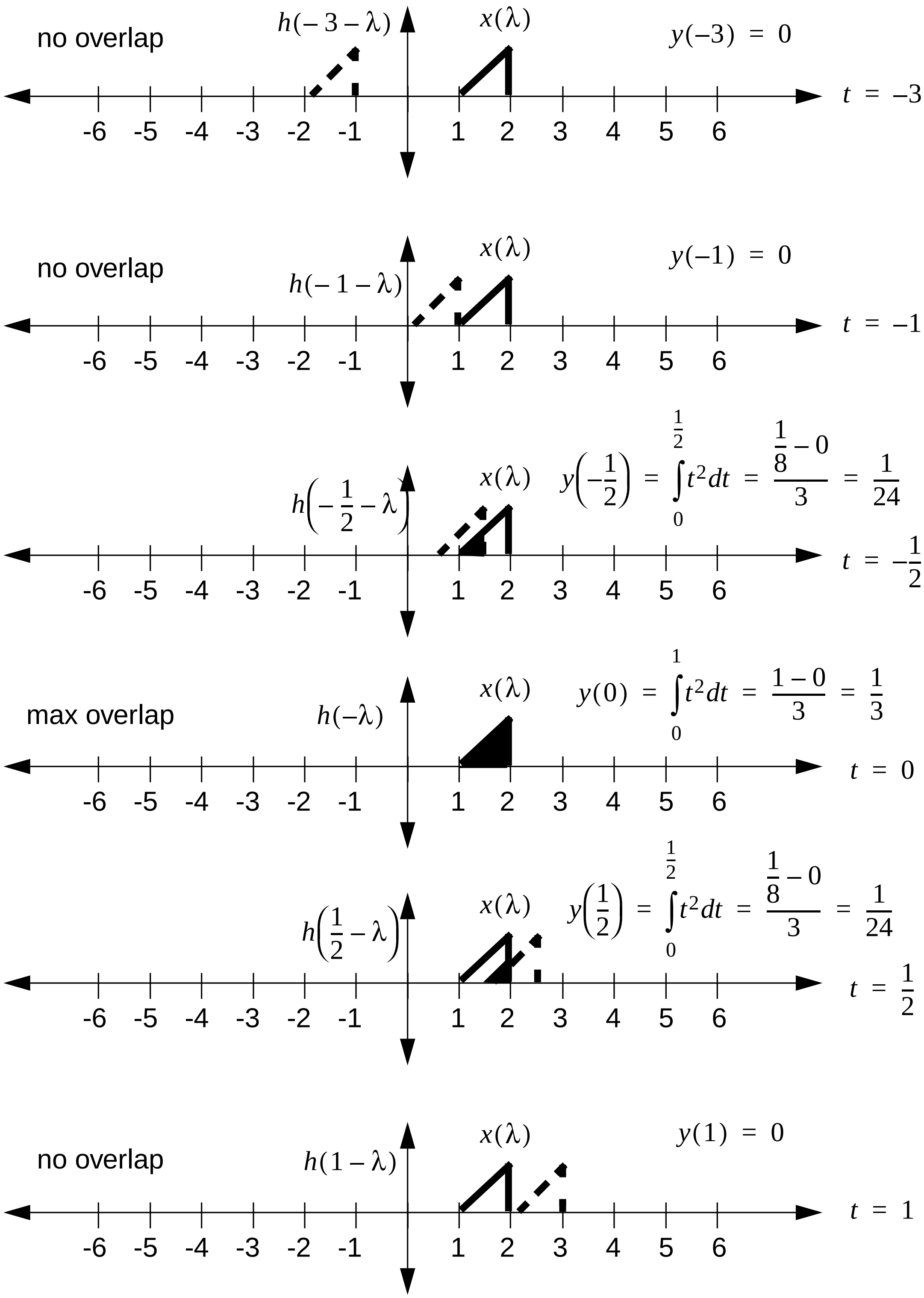


For a more traditional solution, it is necessary to solve the convolution integral,

$y(t) = \int_{-\infty}^{\infty} x(\lambda) \cdot h(t-\lambda) \cdot d\lambda$, at several t and use these points along with observations of the signals trend to plot the signal. For this signal, it seems most appropriate to choose points $t = \left\{-3, -1, -\frac{1}{2}, \frac{1}{2}, 1\right\}$.

From the points taken on the next page the graph below is extrapolated. Notice that the signal starts at $t = -1$, increases in a cubic fashion to a maximum of $\frac{1}{3}$ at $t = 0$, and decreases symmetrically (negative and in a cubic fashion) until $t = 1$.





As a double check, the duration of both the input and the impulse response signals are one, so the convolution of the two signals should have a two second duration, which it does. Also, my graphical solution matches the output of the convolution tool.

(b) Without using the answer to part (a), explain whether the system is causal.

Clearly the signal will not be causal, since the impulse response has negative components. This implies that the output will rely on future inputs.

(c) Use your answer to part (a) to support your reasoning given in (b).

It is very clear from the output given from convolution that the system is non-causal, since the output signal “starts” before the input is applied. The definition of a non-causal input is that the output will anticipate the input, as this system clearly does.

(d) Describe the system using as many system modeling concepts as possible (for example, linearity). Justify your answers.

Since the problem states that the signal can be completely described by its impulse response, and convolution only works for linear, time-invariant (LTI) systems, it must be a linear, time-invariant system.

Problem No. 3: Fourier Series

Given the signal $x(t) = 3\sin(1.5\omega_1 t) + 5\cos(1.75\omega_1 t)$,

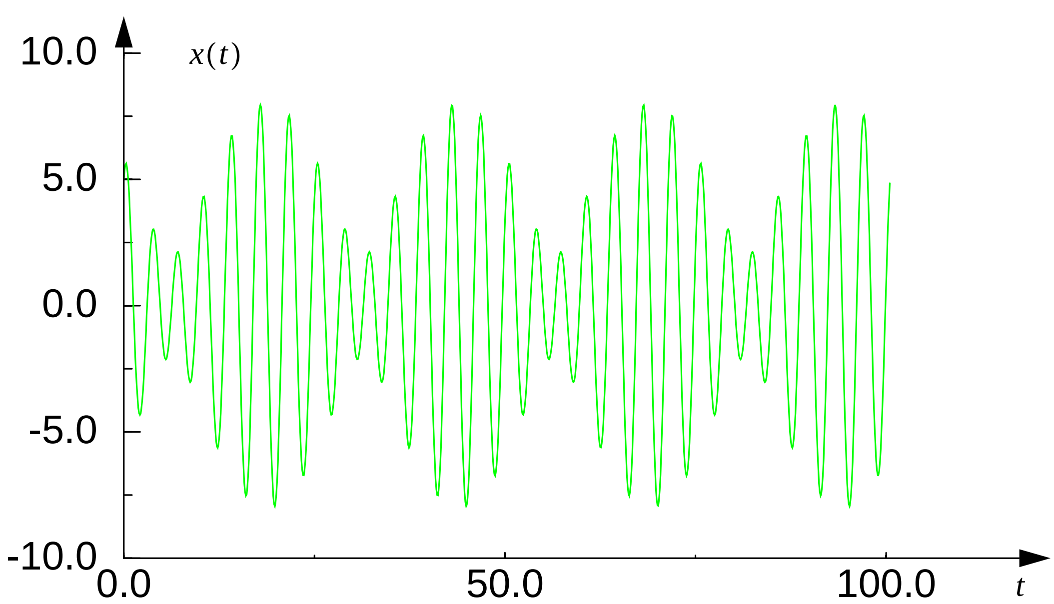
(a) Using symmetry arguments, explain which Fourier coefficients of the trigonometric Fourier Series will be zero ($\{a_n\}$ and $\{b_n\}$). Be careful and be precise:)

Due to linearity, we can break the signal into two components, $x_1(t) = 3\sin(1.5\omega_1 t)$ and $x_2(t) = 5\cos(1.75\omega_1 t)$. Clearly, a sine wave is odd, so $x_1(t)$ is odd ($x_1(t) = -x_1(-t)$). Just as clearly, a cosine wave is even, so $x_2(t)$ is even ($x_2(t) = x_2(-t)$). Since we have both an even and an odd component in the signal, **the signal is neither odd nor even** and no more information about the trigonometric Fourier coefficients can be obtained through symmetry arguments alone.

It will be proven in part (b) that this is the case, since we have both a non-zero a_7 and b_6 .

(b) Compute the Fourier series coefficients.

The first step in computing the Fourier series coefficients is to find the fundamental frequency of the signal. Due to linearity, we can break the signal into two components, $x_1(t) = 3\sin(1.5\omega_1 t)$ and $x_2(t) = 5\cos(1.75\omega_1 t)$. Since the ratio of these frequencies is a rational number, the frequencies are commensurable. Looking at these two frequencies, $f_1 = \frac{1.5\omega_1}{2\pi}$ and $f_2 = \frac{1.75\omega_1}{2\pi}$, without the common terms



of ω_1 and 2π, we get $f_1 = \frac{3}{2}$ and $f_2 = \frac{7}{4}$. Recalling fourth grade math skills, the greatest common denominator for these frequencies is $\frac{1}{4}$. So, the fundamental frequency of the signal $x(t)$ is given by $f_0 = \frac{\omega_1}{8\pi}$, hence $T = 4\left(\frac{2\pi}{\omega_1}\right)$ and $\omega_0 = \frac{\omega_1}{4}$.

Now we move on to finding the Fourier series coefficients.

$$a_0 = \frac{1}{T}\int_0^T x(t)dt = \frac{\omega_1}{8\pi}\int_0^{\frac{8\pi}{\omega_1}} (3\sin(1.5\omega_1 t) + 5\cos(1.75\omega_1 t))dt$$

$$= \frac{3\omega_1}{8\pi}\int_0^{\frac{8\pi}{\omega_1}} \sin(1.5\omega_1 t)dt + \frac{5\omega_1}{8\pi}\int_0^{\frac{8\pi}{\omega_1}} \cos(1.75\omega_1 t)dt$$

$$= \frac{3\omega_1}{8\pi}\left(\frac{-1}{1.5\omega_1}\right)\cos(1.5\omega_1 t)\Big|_0^{\frac{8\pi}{\omega_1}} + \frac{5\omega_1}{8\pi}\left(\frac{1}{1.75\omega_1}\right)\sin(1.75\omega_1 t)\Big|_0^{\frac{8\pi}{\omega_1}}$$

$$= -\frac{1}{4\pi}(\cos(6\pi) - \cos 0) + \frac{5}{14\pi}(\sin(7\pi) - \sin 0) = 0$$

The DC offset value, a_0,is zero as expected, since the original signals could be exactly modeled by normalized sine functions. To find the a and b series of coefficients, we can apply

$$a_n = \frac{2}{T}\int_0^T x(t)\cos(n\omega_0 t)dt$$

and

$$b_n = \frac{2}{T}\int_0^T x(t)\sin(n\omega_0 t)dt.$$

However, in this case the complex mathematics is not necessary. It should be noted that the original signal, a combination of two sinusoids, can be exactly reproduced by a combination of two sinusoids. The only matter is extrapolating the Fourier coefficients by inspection. We need to get an a coefficient to handle the cosine component of the signal ($x_2(t)$) and a b coefficient to handle the sine component of the signal ($x_1(t)$). The sine component needs to occur at an angular frequency of

$$1.5\omega_1 = 1.5(4\omega_0) = 6\omega_0,$$

so the b_6 term will be used to create this component. The cosine component needs to occur at an angular frequency of

$$1.75\omega_1 = 1.75(4\omega_0) = 7\omega_0,$$

so the a_7 term will be used to create this component.

Specifically, **if $a_7 = 5$, $b_6 = 3$, and all other coefficients are zero**, then

$$a_0 + \sum_{n=1}^{\infty} a_n \cos n\omega_0 t + \sum_{n=1}^{\infty} b_n \sin n\omega_0 t = 0 + 5\cos 14\omega_0 t + 3\sin 12\omega_0 t \quad .$$

$$= 5\cos 14\left(\frac{\omega_1}{8}\right)t + 3\sin 12\left(\frac{\omega_1}{8}\right)t = 5\cos 1.75\omega_1 t + 3\sin 1.5\omega_1 t = x(t)$$

Since the function given by the Fourier expansion exactly matches the original signal, we are done.